JOURNAL OF ALGEBRA 116, 342–352 (1988)

A Bound on the Rank of Primitive Solvable Permutation Groups

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Communicated by Gernot Stroth

Received December 23, 1986

1. INTRODUCTION

Let p be a prime and n a positive integer, and let $GF(p^n)$ denote the finite field with p^n elements. Let $S(p^n)$ be the group of all permutations of $GF(p^n)$ of the form $x \to ax^\sigma + b$, for a, b in $GF(p^n)$, $a \neq 0$, and σ in $\mathrm{Aut}(GF(p^n))$. Then $S(p^n)$ is a solvable group of derived length 3 which acts 2-transitively on $GF(p^n)$, and the stabilizer of 0 is $T(p^n) = \{x \to ax^\sigma \mid a \in GF(p^n),\ a \neq 0,\ \sigma \in \mathrm{Aut}(GF(p^n))\}$. Conversely, Huppert [6] showed in 1957 that every finite solvable 2-transitive permutation group is permutation isomorphic to some subgroup of $S(p^n)$, or has one of the degrees 3^2, 5^2, 7^2, 11^2, 23^2, or 3^4. Foulser [4] generalized this to a classification of the finite primitive solvable groups of rank 3 or 4. We generalize this further as follows:

THEOREM 1. *Let $\bar{G}$ be a primitive solvable permutation group of degree d and rank r. Then either*

(i) $d \leqslant ((r + 1.43)/24^{1/3})^c$ *for* $c = 36.435663$ *or*

(ii) *for some integers m, k with $k \leqslant 0.157 \log_3((r + 1.43)/24^{1/3})$, $d = p^{mk}$ and $\bar{G}$ is permutation isomorphic to a solvable subgroup of the wreath product $S(p^m) \operatorname{wr} S_k$ in its primitive action on $(GF(p^m))^k$ (see Section 3; S_k denotes the symmetric group on k elements).*

2. PRIMITIVE SOLVABLE PERMUTATION GROUPS

Let $\bar{G}$ be a primitive solvable permutation group of degree d. Then [6] $d = p^n$ for some prime p, and $\bar{G}$ has a unique minimal normal subgroup V. Moreover, V is regular elementary abelian of order p^n (and so can be regarded as a vector space of dimension n over $GF(p)$), and a one point

0021-8693/88 $3.00

stabilizer $G = \bar{G}_0$, say, of $\bar{G}$ acts on V by conjugation as an irreducible subgroup of the linear group $GL(V) = GL(n, p)$. Conversely [1], if G is an irreducible subgroup of $GL(V)$ then the split extension $\bar{G} = V \cdot G = \{vg \mid v \in V, g \in G$, and $(vg)(\gamma) = g(\gamma) + v$ for $\gamma \in V\}$ is a primitive permutation group acting on V. By definition the rank of $\bar{G}$ is the number of orbits of G in its action on V, and thus the study of the primitive solvable permutation groups of degree $d = p^n$ and rank r reduces to the study of the irreducible solvable subgroups of $GL(n, p)$ with r orbits on the underlying vector space.

So now let G be an irreducible solvable subgroup of $GL(n, p)$. Suppose first that G is primitive as a linear group. Then, improving an earlier qualitative result of Dornhoff [3], we have proved previously [10, Theorem 1]:

THEOREM 2. *Let p be a prime and let G be a solvable primitive subgroup of $GL(n, p)$. Let r be the number of orbits of G on the underlying vector space of p^n elements, and suppose G is not permutation isomorphic to $T(p^n)$ in this action. Then either $r > p^{n/2}/12n + 1$, or p^n is one of* 17^4, 19^4, 7^6, 5^8, 7^8, 11^8, 13^8, 7^9, 3^{16}, *or* 5^{16}.

Thus we need now consider the case where G is an imprimitive linear group. Since a subgroup has at least as many orbits as any group containing it, it will suffice to consider maximal solvable imprimitive linear groups. Such groups can be identified with wreath products, and so we begin by examining wreath product actions.

3. WREATH PRODUCTS

Let H be a subgroup of S_m acting on the set Γ, and K a subgroup of S_n acting on the set Δ. Then the wreath product of H and K is defined to be the group $H \operatorname{wr} K = \{(x_1, ..., x_n; y) \mid x_i \in H, \quad y \in K\}$ with multiplication defined by

$$(x_1, ..., x_n; y)(x'_1, ..., x'_n; y') = (x_{y'(1)}x'_1, ..., x_{y'(n)}x'_n; yy').$$

Then $H \operatorname{wr} K$ has two natural actions (see, for example, [1]):

(i) The imprimitive action on $\Gamma \times \Delta$ defined by $(x_1, ..., x_n; y)(\gamma, \delta) = (x_\delta(\gamma), y(\delta))$. We will denote the permutation group $H \operatorname{wr} K$ acting on $\Gamma \times \Delta$ by $H \operatorname{wr}_I K$. Then $H \operatorname{wr}_I K$ is transitive if H and K both are, and in this case is imprimitive provided K is nontrivial. Further, any maximal imprimitive permutation group is permutation isomorphic to an imprimitive wreath product of two nontrivial permutation groups.

(ii) The primitive action on Γ^n defined by $(x_1, ..., x_n; y)(\gamma_1, ..., \gamma_n) = (\gamma'_1, ..., \gamma'_n)$ for $\gamma'_i = (x_{y^{-1}(i)}(\gamma_{y^{-1}(i)}))$. We will denote the permutation group $H \operatorname{wr} K$ acting on Γ^n by $H \operatorname{wr}_P K$. Then $H \operatorname{wr}_P K$ is transitive provided H is transitive.

Further we have the following:

THEOREM 3 [9, p. 329]. *If H is a transitive permutation group and K is a primitive permutation group which is not regular then $H \operatorname{wr}_P K$ is a primitive permutation group.*

We can combine these wreath product actions as follows:

THEOREM 4 [5, Theorems 1 and 3]. *For any permutation groups H, K, L, we have (up to permutation isomorphism) $(H \operatorname{wr}_I K) \operatorname{wr}_I L = H \operatorname{wr}_I (K \operatorname{wr}_I L)$, and $(H \operatorname{wr}_P K) \operatorname{wr}_P L = H \operatorname{wr}_P (K \operatorname{wr}_I L)$.*

Now note that if Γ can be identified with a linear space U over a finite field so that H acts as a linear group on U, then $H \operatorname{wr}_P K$ acts as a linear group on U^n. This action is in general imprimitive as a linear group:

THEOREM 5 [11, Lemma 15.4]. *Let H be an irreducible subgroup of $GL(U)$ and K a transitive subgroup of S_n, $n > 1$. Then $H \operatorname{wr}_P K$ is an imprimitive linear group on U^n, and is irreducible unless U is a one-dimensional space over $GF(2)$.*

Furthermore these imprimitive linear groups are maximal:

THEOREM 6 [11, Theorem 15.5]. *Let G be a maximal solvable imprimitive subgroup of $GL(n, p)$. Then G is similar as a linear group to a wreath product $H \operatorname{wr}_P K$, where H is a maximal solvable primitive subgroup of $GL(m, p)$, K is a maximal solvable transitive subgroup of S_k, $k > 1$, and $n = mk$.*

The relationship between primitive solvable permutation groups and irreducible solvable linear groups may now be expressed as follows:

THEOREM 7. *Let $\bar{G}$ (resp. $\bar{H}$) be a primitive solvable permutation group with unique minimal normal subgroup V (resp. U) and the stabilizer of 0 denoted by $G \leqslant GL(V)$ (resp. $H \leqslant GL(U)$), so $\bar{G} = V \cdot G$ (resp. $\bar{H} = U \cdot H$). Let K be a transitive solvable subgroup of S_n $(n \geqslant 1)$. Then $\bar{G} = \bar{H} \operatorname{wr}_P K$ if and only if $G = H \operatorname{wr}_P K$ (where equality holds up to permutation isomorphism).*

Proof. If $\bar{G} = \bar{H} \operatorname{wr}_P K$ then $G = \bar{G}_0 = (\bar{H} \operatorname{wr}_P K)_0 = (\bar{H}_0 \operatorname{wr}_P K) = H \operatorname{wr}_P K$ (by the definition of wr_P). Conversely, suppose $G = H \operatorname{wr}_P K$, so

$V = U^n$. Let $\bar{g} \in \bar{G} = V \cdot G$. Then $\bar{g} = vg$ for some v in V, g in G. Since $G = H \operatorname{wr}_P K$, $g = (x_1, ..., x_n; y)$ for some x_i in H, y in K. Since $V = U^n$, $v = (u_1, ..., u_n)$ for some u_i in U. So define $\psi: \bar{G} \to \bar{H} \operatorname{wr}_P K$ by $\psi(\bar{g}) = (u_{y(1)}x_1, ..., u_{y(n)}x_n; y)$, where $u_{y(i)}x_i$ is in $\bar{H} = U \cdot H$. Then for any $\gamma = (\gamma_1, ..., \gamma_n)$ in U^n we have

$$\begin{aligned}
\psi(\bar{g})(\gamma) &= (u_{y(1)}x_1, ..., u_{y(n)}x_n; y)(\gamma_1, ..., \gamma_n) \\
&= ((u_1 x_{y^{-1}(1)})(\gamma_{y^{-1}(1)}), ..., (u_n x_{y^{-1}(n)})(\gamma_{y^{-1}(n)})) \\
&= (x_{y^{-1}(1)}(\gamma_{y^{-1}(1)}) + u_1, ..., x_{y^{-1}(n)}(\gamma_{y^{-1}(n)}) + u_n) \\
&= (x_1, ..., x_n; y)(\gamma_1, ..., \gamma_n) + (u_1, ..., u_n) \\
&= g(\gamma) + v = vg(\gamma) \\
&= \bar{g}(\gamma).
\end{aligned}$$

Thus $\bar{G}$ and $\bar{H} \operatorname{wr}_P K$ are permutation isomorphic in their actions on U^n, which completes the proof.

Finally we consider the number of orbits of the wreath product in its product action:

THEOREM 8. *Let H be a subgroup of S_n acting on the set Γ with h orbits, and let K be a subgroup of S_k, $k \geqslant 1$. Then the number of orbits of $H \operatorname{wr}_P K$ on the set Γ^k is $(1/|K|) \sum_{g \in K} h^{c(g)}$, where for any g in K, $c(g)$ is the number of cycles in the permutation g.*

Proof. Let Ω be the set of orbits of H on Γ, $|\Omega| = h$, and let E be the trivial permutation group on Ω. Then it is easy to see that the number of orbits of $H \operatorname{wr}_P K$ on Γ^k is equal to the number of orbits of $E \operatorname{wr}_P K$ on Ω^k. By [7, Theorem 4.10] this number is $(1/|K|) \sum_{g \in K} |\Omega|^{c(g)} = (1/|K|) \sum_{g \in K} h^{c(g)}$.

As a consequence of this theorem, for any positive integers h, k and $K \leqslant S_k$ we may define $(h)K$ to be the number of orbits of $H \operatorname{wr}_P K$ for any group H having h orbits, and so

$$(h)K = \frac{1}{|K|} \sum_{g \in K} h^{c(g)}. \tag{1}$$

Note that $(1)K = 1$ for any K, and so we are interested in the case $h \geqslant 2$.

4. PRIMITIVE COMPONENTS

We now find the primitive components of those solvable groups having the least number of orbits for any fixed degree.

Following the notation of Pálfy [8], we let $R_q = S(q)$ for any q prime, so $R_q = \{x \to ax + b \mid a, b \in GF(q), a \neq 0\}$, considered as a subgroup of S_q, and let $R_4 = S_4$ $(= S(2^2))$. Then R_k is a primitive permutation group which is the unique maximal solvable transitive subgroup of S_q. Further, $|R_q| = q(q-1)$ for q prime, $|R_4| = 24$, and $R_q = S_q$ for $q = 2, 3, 4$.

Now for $q \geqslant 5$, R_q has one element with q cycles, $q-1$ elements with one cycle, and for each k dividing $q-1$, $\phi(k)q$ elements with $1 + (q-1)k$ cycles, whee $\phi(k)$ is the Euler ϕ-function. Thus if H has h orbits, the number of orbits of $H \operatorname{wr}_P R_q$ is

$$(h)\, R_q = \left(h^q + (q-1)h - \sum_{\substack{k \mid (q-1) \\ k \neq 1}} q\phi(k)\, h^{(1 + (q-1)/k)} \right) \Big/ q(q-1). \tag{2}$$

Furthermore, Foulser [4, Lemma 2.6] proved that $(h)\, S_k = \binom{k+h-1}{k}$ and so

$$(h)\, R_q = \binom{q+h-1}{q} \qquad \text{for} \quad q = 2, 3, 4. \tag{3}$$

Now suppose K is a maximal solvable primitive subgroup of S_8. Since the stabilizer of 0 in K must be a maximal solvable irreducible subgroup of $GL(3, 2)$ the only possibility for K [11, Theorem 21.6] is $K = S(2^3)$ of order 168. Therefore let $R_8 = S(2^3)$. By examining the cycle structure of $S(2^3)$ we have via (1):

$$(h)\, R_8 = (h^8 + 63h^4 + 104h^2)/168. \tag{4}$$

Finally suppose K is a maximal solvable primitive subgroup of S_9. Since the stabilizer of 0 in K must be a maximal solvable irreducible subgroup of $GL(2, 3)$, the only possibility for K [11, Theorem 21.6] is the 2-transitive group of degree 3^2 described by Huppert [6], with stabilizer $GL(2, 3)$ and order 432; we denote this group by R_9 (note that in this case, $S(3^2)$ is properly contained in R_9). Then by examining the cycle structure of R_9 we have

$$(h)\, R_9 = (h^9 + 36h^6 + 33h^5 + 182h^3 + 180h^2)/432. \tag{5}$$

Thus for each $k = 4, 8, 9$ or a prime, we have determined the exact formula for $(h)\, R_k$, where R_k is the (essentially) unique maximal solvable primitive permutation group of degree k. But for any k_1, $k_2 = 4, 8, 9$ or a prime, if H has h orbits then by definition $H \operatorname{wr}_P R_{k_1}$ has $(h)\, R_{k_1}$ orbits, and so $H \operatorname{wr}_P (R_{k_1} \operatorname{wr}_I R_{k_2}) = (H \operatorname{wr}_P R_{k_1}) \operatorname{wr}_P R_{k_2}$ has $(h)(R_{k_1} \operatorname{wr}_I R_{k_2}) = ((h)\, R_{k_1})\, R_{k_2}$ orbits. Thus we can calculate the exact number of orbits for strings of wreath products of these components. We now prove that the R_k defined above are sufficient to calculate the least number of orbits possible for $H \operatorname{wr}_P K$, for H having h orbits, $h \geqslant 2$, and K a transitive solvable subgroup of S_k. Indeed, if K is imprimitive, we can write $K \leqslant (K_1 \operatorname{wr}_I \cdots \operatorname{wr}_I K_t)$, where each K_i is a primitive subgroup of S_{k_i} with $k = k_1 k_2 \cdots k_t$, so it will suffice to prove the following theorem:

THEOREM 9. *Let K be a maximal solvable primitive subgroup of S_k, where $k = q^s$, q prime, $s > 1$. Then there exist transitive solvable subgroups K_i of S_{k_i}, $i = 1, 2$, such that $k = k_1 k_2$ and $(h)(K_1 \operatorname{wr}_I K_2) < (h)K$ for all $h \geqslant 2$, except when $k = 4$, or for the pairs $(h, k) = (2, 8)$, $(3, 8)$, $(4, 8)$, $(2, 9)$, $(3, 9)$, $(4, 9)$, $(5, 9)$, $(6, 9)$.*

Proof. Let $R_q^{(i)}$ denote $R_q \operatorname{wr}_I R_q \operatorname{wr}_I \cdots \operatorname{wr}_I R_q$ (i factors) for $i > 1$. We will show $(h)(R_q^{(s)}) < (h)K$ for $h \geqslant 2$, $q > 2$. First note that by Pálfy [8, Theorem 1], the order of a primitive solvable permutation group T of degree $d = q^t$ is at most

$$|T| \leqslant 24^{-1/3} d^{c_q}, \tag{6}$$

where $c_2 = 1 + \log_4(6 \cdot 24^{1/3}) = 3.05664...$, $c_3 = 1 + \log_9(48 \cdot 24^{1/3}) = 3.24399...$, $c_5 = 1 + \log_{25}(96 \cdot 24^{1/3}) = 2.74710...$, $c_7 = 1 + \log_{49}(144 \cdot 24^{1/3}) = 2.54918...$; and $c_q = 1 + \log_q((q-1) \cdot 24^{1/3}) < 2.5$ for $q > 7$. Thus since the identity element in S_k has k cycles, (1) and (6) imply

$$(h)K > \frac{1}{|K|} h^k \geqslant 24^{1/3} h^k / (k^{c_q}). \tag{7}$$

Now $(h)\, R_q$ is a polynomial in h of degree q with all coefficients non-negative, and so if we let $d_{q,h} = h^q / (h)\, R_q$ then $d_{q,h}$ increases with h for q fixed and $h > 0$, and in particular $((h)\, R_q)\, R_q = ((h)\, R_q)^q / d_{q,(h)R_q} \leqslant ((h)\, R_q)^q / d_{q,h} = h^{q^2} / d_{q,h}^{q+1}$. Thus by induction we get

$$(h)\, R_q^{(s)} < h^{q^s} / d_{q,h}^{((q^s - 1)/(q-1))}. \tag{8}$$

Combining (7) and (8) we see that the theorem holds whenever

$$h^{q^s} / d_{q,h}^{((q^s - 1)/(q-1))} \leqslant 24^{1/3} h^{q^s} / (q^s)^{c_q} \tag{9}$$

or, equivalently,

$$q^{sc_q} \leqslant 24^{1/3} (d_{q,h})^{((q^s - 1)/(q-1))}. \tag{10}$$

Now for $q \geqslant 11$, (2) shows that $(h)\, R_q < 2h^q / q(q-1)$ and so $d_{q,h} \geqslant q(q-1)/2$ for all $q \geqslant 11$, $h \geqslant 2$. Thus (10) is satisfied for $q \geqslant 11$ if $q^{sc_q} \leqslant 24^{1/3} (q(q-1)/2)^{((q^s - 1)/(q-1))}$ which certainly holds if $2.5s \leqslant 11^{s-1}$. Hence the theorem holds for all $q \geqslant 11$, $s, h \geqslant 2$. Now using formulas (2) and (3) we can calculate $d_{q,h}$ exactly for small values of q as follows:

q	h	$d_{q,h}$
7	2	64/5
5	2	16/3
3	2	2
3	11	121/26
2	2	4/3
2	5	5/3

By combining each of these with (10) we can see that the theorem now holds for all cases except possibly $q=3$, $s=2, 3$, $h=2, 3, ..., 10$; $q=2$, $s=4, 5$, $h=2, 3, 4, 5$; and $q=2$, $s=2, 3$, $h \geqslant 2$.

Now by (3), $(h)\, R_3 = \binom{h+2}{3}$ and so comparing $(h)(R_3^{(3)}) = (((h)\, R_3)\, R_3)\, R_3$ with (7) for $k=3^3$ and $h=2, 3, ..., 10$ proves the result for $q=3$, $s=3$. For $q=3$, $s=2$ a comparison of (3) and (5) shows $(h)\, R_q < (h)(R_3^{(2)}) = ((h)\, R_3)\, R_3$ if and only if $h=2, 3, 4, 5, 6$. Similarly for $q=2$, $s=5$, combining (3) and (7) gives $(h)(R_4 \operatorname{wr}_I R_4 \operatorname{wr}_I R_2) < (h)K$ for all $h \geqslant 2$.

Next consider $q=2$, $s=4$. Then K is maximal solvable primitive of degree 2^4, and so the stabilizer of 0 in K must be a maximal solvable irreducible subgroup of $GL(4, 2)$. Thus the only possibility for K [11, Theorem 18.5, Lemma 21.2] is $S(2^4)$ of order 960 or $S(2^2) \operatorname{wr}_P S_2$ of order 1152. Therefore by (1), $(h)K > h^k/1152$. Now combining this with (3) and (4) gives $(h)(R_4 \operatorname{wr}_I R_4) < (h)K$ for $h=3, 4, 5$, and $(2)(R_8 \operatorname{wr}_I R_2) < (2)K$.

Finally using (3) and (4) we have $(h)\, R_4 < (h)(R_2 \operatorname{wr}_I R_2)$ for all $h \geqslant 2$ and $(h)\, R_8 < (h)(R_2 \operatorname{wr}_I R_4)$ $(\leqslant (h)(R_4 \operatorname{wr}_I R_2))$ if and only if $h=2, 3, 4$. This completes the proof of the theorem.

5. A Bound on the Number of Orbits

Now define $r(k, h) = \min\{(h)K \mid K$ is a transitive solvable subgroup of $S_k\}$ for $h \geqslant 2$, $k \geqslant 1$. Then applying Theorems 9 and 4 gives $r(k, h) = (h)(R_{k_1} \operatorname{wr}_I \cdots \operatorname{wr}_I R_{k_s})$ for some $k_1, ..., k_s = 4, 8, 9$ or a prime, such that $k_1 k_2 \cdots k_s = k$, and so $r(k, h) = r(k_s, r(k_{s-1}, ..., r(k_1, h) \cdots))$. Since we have an exact formula for $r(k_i, h) = (h)\, R_{k_i}$ for all such k_i, we could in theory calculate $r(k, h)$ exactly for any $k \geqslant 1$, $h \geqslant 2$. We now try to bound $r(k, h)$ in general. First we note that the value of k_i giving proportionately the least number of orbits is 4 for $h \geqslant 4$ and 8 for $h=2, 3$. Let $f(h) = (h)\, R_4 = \binom{h+3}{4}$. Then since $r(2, r(4, h)) \leqslant r(4, r(2, h))$ and $r(2, r(2, h)) \geqslant r(4, h)$ for all $h \geqslant 2$

(by (3)), it follows from Theorem 9 that $r(4^i, h) = f^i(h)$ for all $h \geqslant 4$, and so $r(k, h) = f^{\log_4 k}(h)$ whenever $h \geqslant 4$ and k is a power of 4. Similarly, $r(k, h) = f^{\log_4(k/8)}(h)$ for $h = 2, 3$ whenever $k/8$ is a power of 4. In order to develop this into a formula applicable to all k, we approximate $f(h)$ by a more amenable formula: $f(h) = \binom{h+3}{4} = (h^4 + 6h^3 + 11h^2 + 6h)/24$ and so $f(h) > (h + 1.43)^4/24 - 1.43$ for all $h \geqslant 4$. Thus we have

$$\begin{aligned} f^s(h) &\geqslant (h + 1.43)^{4^s}/(24^{(4^s - 1)/3}) - 1.43 \\ &= 24^{1/3}((h + 1.43)/24^{1/3})^{4^s} - 1.43 \end{aligned}$$

for all $h \geqslant 4$. So we define the function

$$\bar{r}(y, x) = 24^{1/3}((x + 1.43)/24^{1/3})^y - 1.43 \qquad \text{for all} \quad x, y > 0.$$

Then note that $\bar{r}(y_1 y_2, x) = \bar{r}(y_2, \bar{r}(y_1, x))$ for all $x, y_1, y_2 > 0$. To allow for the effect of $r(8, h)$ when $h = 2$ or 3 we define

$$x_h = \begin{cases} \bar{r}(1/8, r(8, h)) & \text{for } h = 2, 3 \\ h & \text{otherwise} \end{cases}$$

so $x_2 = 1.9962...$, $x_3 = 2.9148...$, and $\bar{r}(k, x_h) = \bar{r}(k/8, r(8, h))$, $h = 2, 3$. Then we have the following theorem:

THEOREM 10. *$r(k, h) \geqslant \bar{r}(k, x_h)$ for all $k \geqslant 1$, $h \geqslant 2$.*

Proof. Since $r(k, h) = r(k_s, r(k_{s-1}, ..., r(k_1, h)...))$ for some $k_1, ..., k_s = 4$, 8, 9, or a prime, such that $k_1 k_2 \cdots k_s = k$ we use induction on s.

Suppose first that $s = 1$. If $k = 4$ and $h \geqslant 4$ then $\bar{r}(4, x_h) = (h + 1.43)^4/24 - 1.43 < f(h) = r(4, h)$, and for $h = 2, 3$, $\bar{r}(4, x_h) < r(4, h)$ by direct calculation using (3). Similarly using (4) and (5) we have $\bar{r}(8, x_h) < r(8, h)$, $h = 2, 3, 4$ and $\bar{r}(9, x_h) < r(9, h)$, $h = 2, 3, 4, 5, 6$, and so by Theorem 9 we may now assume $k = q$, q prime. Then by (2) and (3), $r(q, h) > h^q/q(q-1)$ and so the theorem holds if

$$24^{1/3} q(q-1) < (24^{1/3} h/(h + 1.43))^q. \tag{11}$$

This is true for all cases except $q \leqslant 11$, $h = 2$; $q \leqslant 5$, $h = 3, 4, 5$; $q = 3$, $h = 6, 7, ..., 12$; and $q = 2$, $h = 6, 7$. But then exact calculations using (2) and (3) show that $\bar{r}(q, x_h) < r(q, h)$ in each of these cases, and thus the theorem holds for $s = 1$.

Now suppose that $s > 1$. Note that for k fixed, $\bar{r}(k, x)$ is an increasing function of x for $x > 1$. Thus if $s = 2$, $k_1 = 2$ and $h = 2$ then since $\bar{r}(2, x_2) < x_3$ we have $\bar{r}(k, x_h) = \bar{r}(k_2, \bar{r}(2, x_2)) < \bar{r}(k_2, x_3) \leqslant r(k_2, 3) =$

$r(k_2, r(2, 2)) = r(k, 2)$. Otherwise, for $\bar{h} = r(k_{s-1} \cdots k_1, h)$ we have $\bar{h} \geq 4$ and $x_{\bar{h}} = \bar{h}$ and so

$$\begin{aligned}
\bar{r}(k, x_h) &= \bar{r}(k_s, \bar{r}(k_{s-1} \cdots k_1, x_h)) \\
&\leq \bar{r}(k_s, r(k_{s-1} \cdots k_1, h)) \quad \text{(by induction)} \\
&= \bar{r}(k_s, \bar{h}) \\
&= \bar{r}(k_s, x_{\bar{h}}) \\
&\leq r(k_s, \bar{h}) \quad \text{(by induction)} \\
&= r(k_s, r(k_{s-1} \cdots k_1, h)) \\
&\leq r(k_s k_{s-1} \cdots k_1, h) \\
&= r(k, h).
\end{aligned}$$

This completes the proof of the theorem.

6. The Proof of Theorem 1

Let $\bar{G}$ be a primitive solvable permutation group of degree $d = p^n$ and rank r. As before we may assume $\bar{G}$ is maximal. Let $G = \bar{G}_0$ be the stabilizer of 0 in G considered as a maximal solvable irreducible subgroup of $GL(n, p)$. Then by Theorem 6 we may write G in the form $G = H \operatorname{wr}_P K$, where H is a maximal solvable primitive subgroup of $GL(m, p)$, K is a maximal solvable transitive subgroup of S_k for some $k \geq 1$, and $n = mk$.

Suppose first that H is permutation isomorphic to $T(p^m)$. Then $G = (S(p^m))_0 \operatorname{wr}_P K$, and so by Theorem 7, $\bar{G} = S(p^m) \operatorname{wr}_P K$. But $T(p^m)$ has 2 orbits, and so the number of orbits of $G = T(p^m) \operatorname{wr}_P K$ is $r = (2)K \geq r(k, 2) \geq \bar{r}(k, x_2) = \bar{r}(k/8, 10) = 24^{1/3}(11.43/24^{1/3})^{k/8} - 1.43$. Thus $(r + 1.43)/24^{1/3} \geq ((11.43/24^{1/3})^{1/8})^k$, and so $k \leq 0.157 \log_3((r + 1.43)/24^{1/3})$.

Now suppose H is not permutation isomorphic to $T(p^m)$. Then we want to show $d \leq ((r + 1.43)/24^{1/3})^c$ for $c = 36.435663$, or

$$r \geq 24^{1/3} d^{1/c} - 1.43. \tag{12}$$

Let h be the number of orbits of H. Then $r = (h)K \geq r(k, h) \geq \bar{r}(k, x_h)$ by Theorem 10, and so the theorem holds if $\bar{r}(k, x_h) \geq 24^{1/3} d^{1/c} - 1.43$ or $24^{1/3}((x_h + 1.43)/24^{1/3}) - 1.43 \geq 24^{1/3} d^{1/c} - 1.43$. But $d = p^{mk}$, and so the theorem holds if $((x_h + 1.43)/24^{1/3})^k \geq p^{mk/c}$, or

$$p^m \leq ((x_h + 1.43)/24^{1/3})^c. \tag{13}$$

Now Huppert [6] proved that for $h=2$, $p^m \leqslant 23^2$, and Foulser [4] showed that for $h=3$, $p^m \leqslant 7^4$ and for $h=4$, $p^m \leqslant 3^{10}$; all of these satisfy (13). So we may assume $h \geqslant 5$ (and so $x_h = h$). But $((5+1.43)/24^{1/3})^c > 4.8 \times 10^{12}$, and so we may assume $p^m > 4.8 \times 10^{12}$.

Now since H is a maximal solvable primitive linear group, by Theorem 2 $h > p^{m/2}/12m$, with 10 possible exceptions for p^m all less than 4.8×10^{12}. Thus (13) holds if $p^m \leqslant ((p^{m/2}/12m + 1.43)/24^{1/3})^c$ and so the theorem holds if

$$(p^m)^{(1/2-(1/c))} \geqslant 24^{1/3} 12m. \tag{14}$$

Since $p^m > 4.8 \times 10^{12}$ and $\frac{1}{2} - (1/c) = 0.47255...$, (14) holds whenever $m < (4.8 \times 10^{12})^{0.47255}/(24^{1/3}12) = 28396.7...$. So we may assume $m > 28396$. But $p \geqslant 2$, and so (14) holds whenever $2^{0.47255m} \geqslant 24^{1/3}12m$, which is certainly true for $m > 28396$. This completes the proof of the theorem.

Note. The bound $d \leqslant ((r+1.43)/24^{1/3})^c$ is almost asymptotic. For, let $\bar{H}$ be the maximal 2-transitive permutation group of degree 23^2 described by Huppert [6]. Let $\bar{G}_i = (\bar{H} \operatorname{wr}_P R_8) \operatorname{wr}_P R_4^{(i)}$ for $i \geqslant 1$. Then $\bar{G}_i$ has degree $d_i = 23^{16 \cdot 4^i}$ and (by (3) and (4)) rank $r_i = f^i(10)$ for $f(x) = \binom{x+3}{4} < 24^{1/3}((x+1.5)/24^{1/3})^4 - 1.43$ for all $x \geqslant 10$, and so for $c' = 36.275 < c = 36.435663$, we have $((r_i + 1.43)/24^{1/3})^{c'} \leqslant d_i \leqslant ((r_i + 1.43)/24^{1/3})^c$ for all $i \geqslant 1$.

The solvable subgroups of $S(p^m) \operatorname{wr}_P S_k$ of low rank are difficult to determine; those for the Huppert case of $k=1$ can be found in [12]. However, it is possible to bound the derived length of such groups as follows:

COROLLARY. *Let $\bar{G}$ be a primitive solvable permutation group of degree d and rank r. Let $s(\bar{G})$ be the derived length of $\bar{G}$. Then either $d \leqslant ((r+1.43)/24^{1/3})^c$ or $s(\bar{G}) \leqslant 7.2161 + 5/2 \log_3 \log_3((r+1.43)/24^{1/3})$.*

Proof. Dixon [2] proved that the derived length of a solvable permutation group K of degree d is at most $s(K) \leqslant 5(\log_3 d)/2$. Since $s(\bar{H} \operatorname{wr} K) \leqslant s(\bar{H}) + s(K)$ and $s(S(p^m)) = 3$, the result follows from this and Theorem 1.

ACKNOWLEDGMENTS

The work involved in this paper was part of my Ph.D. thesis at Carleton University. I thank my supervisor, Dr. J. D. Dixon, for his help throughout.

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